MOD-Y

1)-> Types of Random Variable:-

=> Bernoulli's Random Variable ->

Also called Bernoulli's Trials. Suppose in a trial, outcomes can be classified as either the success or a failure. Let (X) = 1, when the outcome is a success & (X) = 0, when the outcome is a failure. The probability mass functn:

$$\boxed{\begin{aligned} P(1) &= P\{X = 1\} = p \\ P(0) &= P\{X = 0\} = (1-p) \end{aligned}} \quad (1 \text{ outcome})$$

where, (p) is the probability that the trial is a success.

A random variable (X) is said to be

Bernoulli's Random Variable, if its probability mass funct^n is given by as above.

=> Binomial Random Variable ->

Suppose, (n) independent trials, each of which results in a success with probability (P) or in a failure with probability (1 - p) are to be performed.

If (X) represents the no. of success that occurs in the (n) trials, then (X) is said to be a Binomial Random Variable with parameter (n, p).

Thus, a Bernoulli's Random Variable is just a Binomial Random Variable with parameter (1, p).

The probability mass funct^n of a binomial random variable having parameter (n, p) is given by: (Finite Outcomes)

$$\therefore \boxed{P(i) = \binom{n}{i} p^{i} \cdot (1-p)^{n-i} = {}^{n}C_{i}\, p^{i} \cdot (1-p)^{n-i}}$$

Q: A pair of dice is thrown 7 times. If getting a total of 7 is success, what is the probability of:

ⓐ No success [(1,6) (2,5) (3,4) (4,3) (5,2) (6,1)]

ⓐ 6 success

ⓐ Atleast 6 success

ⓐ Atmost 6 success

Ans - Given, n = 7

Let, X = Success (getting a total of 7)

$\therefore P(\text{success}) = \frac{6}{36} = \frac{1}{6}$

$\therefore P(\text{failure}) = 1 - \frac{1}{6} = \frac{5}{6}$

∴ No. of success = 0, 1, 2, 3, 4, 5, 6, 7 (within 7 throws)

ⓐ $P\{X = 0\}$ i.e. P(no. of success)

$= {}^{7}C_{0}\, p^{0} (1-p)^{7-0} = \left(\frac{5}{6}\right)^{7} = 0.279$ (27.9%)

ⓐ $P(6 \text{ success}) = P\{X = 6\} = P(6)$

$= {}^7C_6 \left(\frac{1}{6}\right)^6 \left(\frac{5}{6}\right)^1 = 0.000125$ (0.0125%)

ⓐ $P(\text{atleast } 6 \text{ success}) = P\{X \geq 6\}$

$= P\{X = 6\} + P\{X = 7\}$

$= {}^7C_6 \left(\frac{1}{6}\right)^6 \left(\frac{5}{6}\right)^1 + {}^7C_7 \left(\frac{1}{6}\right)^7 \left(\frac{5}{6}\right)^0 = 0.0001286$ (0.0129%)

ⓐ $P(\text{atmost } 6 \text{ success}) = P\{X \leq 6\}$

$= 1 - P\{X > 6\} = 1 - P\{X = 7\}$

$= 1 - {}^7C_7 \left(\frac{1}{6}\right)^7 \left(\frac{5}{6}\right)^0 = 0.99$ (99.99%)

Q. $\sum_{i=0}^{n} P(i) = 1$

Ans - Proof: $\sum_{i=0}^{n} P(i) = \sum_{i=0}^{n} {}^nC_i \, p^i (1-p)^{n-i} = \sum_{i=0}^{n} (p+1-p)^n$

$= 1$ (HP)

Q: 5 fair coins are flipped. If the outcomes are assumed independent, find the probability mass functn of the no. of heads obtained.

Ans - Given, $n = 5$.

$\therefore$ Let $X = \{\text{No. of heads obtained}\}$

$= \{0, 1, 2, 3, 4, 5\}$

$\therefore P\{X = 0\} = P(0) = {}^5C_0 (1/2)^0 (1/2)^5 = 1/32$

$\therefore P\{X = 1\} = P(1) = {}^5C_1 (1/2)^1 (1/2)^4 = 5/32$

$\therefore P\{X = 2\} = P(2) = {}^5C_2 (1/2)^2 (1/2)^3 = 10/32$

$\therefore P\{X = 3\} = P(3) = {}^5C_3 (1/2)^3 (1/2)^2 = 10/32$

$\therefore P\{X = 4\} = P(4) = {}^5C_4 (1/2)^4 (1/2)^1 = 5/32$

$\therefore P\{X = 5\} = P(5) = {}^5C_5 (1/2)^5 (1/2)^0 = 1/32$

// → Properties of Binomial Random variable:-

⇒ Expected value or Mean of Binomial Random variable:-

$E[X] = \sum_{i=0}^{n} i \cdot P(i) = \sum_{i=0}^{n} i \binom{n}{i} p^i (1-p)^{n-i}$

$= 0 + \binom{n}{1} p (1-p)^{n-1} + 2\binom{n}{2} p^2 (1-p)^{n-2}$

$+ 3\binom{n}{3} p^3 (1-p)^{n-3} + \ldots + (n-1)\binom{n}{n-1} p^{n-1}$

$\cdot (1-p) + n \cdot p^n$

$= np(1-p)^{n-1} + 2 \cdot \frac{n(n-1)}{2!} p^2 (1-p)^{n-2}$

$+ 3 \frac{n(n-1)(n-2)}{3!} \cdot p^n (1-p)^{n-3} + \ldots + (n-1) np^{n-1}$

$\cdot (1-p) + np^n$

$= np\left[(1-p)^{n-1} + (n-1)p(1-p)^{n-2} + \frac{(n-1)(n-2)}{2!}\right.$

$\left. \cdot p^2 (1-p)^{n-2} + \ldots + (n-1)p^{n-2}(1-p) + p^{n-1}\right]$

$= np\left[(1-p)^{n-1} + (n-1)p(1-p)^{(n-1)-1} + \frac{(n-1)(n-2)}{2!}\right.$

$\cdot p^2 (1-p)^{(n-1)-2} + \ldots + (n-1)p^{(n-1)-1}(1-p)$

$\left. + p^{n-1}\right]$

$= np \sum_{i=0}^{n-1} \binom{n}{i} p^i (1-p)^{n-i} = np[p + (1-p)]^{n-1}$

$= np \cdot 1 = np$

$\therefore \boxed{E[X] = np}$

11 → Find $E[X^K]$ for Binomial Random Variable: —

$E[X^K] = \sum_{i=0}^{n} i^K p(i) = \sum_{i=0}^{n} i^K \binom{n}{i} p^i (1-p)^{n-i}$

$= \sum_{i=0}^{n} i^{K-1} \cdot i \binom{n}{i} p^i (1-p)^{n-i}$

$= \sum_{i=0}^{n} i^{K-1} n \binom{n-1}{i-1} p^i (1-p)^{n-i}$

$= np \sum_{i=1}^{n} i^{K-1} \binom{n-1}{i-1} p^{i-1} (1-p)^{n-i}$

$= np \sum_{j=0}^{n-1} (j+1)^{K-1} \binom{n-1}{j} p^j (1-p)^{(n-1)-j}$

$$\boxed{= np\cdot E[(Y+1)^{k-1}] = E[X^k]}$$

$$\therefore E[X^2] = np\,E[(Y+1)] = np[E[Y]+1]$$

$$= np[\{(n-1)p\}+1]$$

$$\therefore Var(X) = E[X^2] - (E[X])^2$$

$$= np[(n-1)p+1] - (np)^2$$

$$= np(np-p+1) - (np)^2$$

$$= \cancel{(np)^2} - np^2 + np - \cancel{(np)^2} = np(1-p)$$

$$\therefore \boxed{E[X] = np \;\&\; Var(X) = np(1-p)}$$

$$\therefore \boxed{Var(X) = (1-p)\cdot E[X]}$$

Q: In a hurdle race, a player has to cross 10 hurdles. The probability of clearing a hurdle is 5/6. What is the probability that he will knock fewer than 2 hurdles.

Ans Here, $n = 10$

$\therefore P(\text{clearing a hurdle}) = 5/6 = (1-p)$

$\therefore P(\text{knocking a hurdle}) = 1 - 5/6 = 1/6 = (p)$

$$\therefore P\{X<2\} = P\{X=0\} + P\{X=1\}$$

$$= {}^{10}C_0 \left(\frac{1}{6}\right)^0 \cdot \left(\frac{5}{6}\right)^{10} + {}^{10}C_1 \left(\frac{1}{6}\right)^1 \left(\frac{5}{6}\right)^9 = 0.485$$

(48.5%)

Q: How many times must a man toss a ~~coin~~ fair coin, so that probability of having atleast 1 head is more than 80%.

Ans- Let X: (No. of heads)

$$\therefore P\{X \geq 1\} > 80\% = 4/5$$

$$\Rightarrow 1 - P\{X<1\} > \frac{4}{5}$$

$$\Rightarrow 1 - P\{X=0\} > \frac{4}{5}$$

$$\Rightarrow 1 - {}^{n}C_0 \left(\frac{1}{2}\right)^0 \left(\frac{1}{2}\right)^n > \frac{4}{5}$$

$$\Rightarrow 1 - \left(\frac{1}{2}\right)^n > \frac{4}{5} \Rightarrow 1 - \frac{4}{5} > \left(\frac{1}{2}\right)^n$$

$\Rightarrow \left(\frac{1}{2}\right)^n < \frac{1}{5} \Rightarrow \frac{1}{2^n} < \frac{1}{5} \Rightarrow 5 < 2^n \Rightarrow 2^n > 5$

$\Rightarrow n \geq 3$ [So, the man must toss the fair coin atleast 3 times.]

So, the man must toss the fair coin more than or equal to 3 times.

II-> **Binomial distributⁿ functⁿ:-**

Let (X) be a binomial random variable with parameter (n, p), then its distributⁿ functⁿ is defined by:

$$\boxed{P\{X \leq i\} = \sum_{K \leq i} {}^{n}C_{K}\, p^{K}(1-p)^{n-K}}, \quad i \in W$$

II-> **Poisson's Random variable:** (Discrete random variable)

A random variable (X), that takes on 1 of the value 0, 1, 2, 3, 4, . . . is said to be a Poisson's random variable with parameter (λ), if for some λ > 0: (Uncountable outcomes)

$$\boxed{P(i) = P\{X = i\} = \frac{e^{-\lambda}\lambda^{i}}{i!}} \quad \text{--- ①}$$

Eqⁿ ① defines a probability mass functⁿ, since $\sum P(i) = \sum_{i=0}^{\infty} \frac{e^{-\lambda}\lambda^{i}}{i!} = e^{-\lambda} \cdot \sum_{i=0}^{\infty} \frac{\lambda^{i}}{i!} = e^{-\lambda} \cdot e^{\lambda} = 1$

In words, [a Poisson's random variable is a count of the no. of occurrences of an event in a given unit of time or distance or area or volume.]

⇒ Conditⁿs:

ⓐ Suppose, events are occurring independently.

ⓐ The probability that an event occurs in a given length of time doesn't change through time.

Then (X): the no. of events in a fixed unit of time has the poisson's distributⁿ:

$$\boxed{P(i) = P\{X = i\} = \frac{e^{-\lambda}\lambda^{i}}{i!}}$$

where (λ) is the mean no. of events.

Ex: ⓐ The no. of misprints on a page of a book.

ⓐ The no. of people in a community, who survive to age 100.

ⓐ The no. of wrong telephone no. that are dialed in a day.

Q: Suppose that the no. of typographical errors on a single page of a book has the Poisson's distributn with parameter $\lambda = 0.5$. Calculate the probability that there is atleast 1 error on this page.

Ans- X: no. of typographical error.

$$\therefore P\{X \geq 1\} = 1 - P\{X < 1\} = 1 - P\{X = 0\}$$

$$= 1 - \frac{e^{-\lambda}\lambda^0}{0!} = 1 - e^{-0.5} = 0.39 = 1 - \frac{1}{\sqrt{2.718}}$$

II -> **Expected value of poisson's random variable:-**

$$\therefore E[X] = \sum_{i=0}^{\infty} i\, P(i) = \sum_{i=0}^{\infty} i \cdot \frac{e^{-\lambda}\lambda^i}{i!}$$

$$= e^{-\lambda} \cdot \sum_{i=1}^{\infty} \frac{\lambda^i}{(i-1)!} = e^{-\lambda} \cdot \lambda \cdot \sum_{i=1}^{\infty} \frac{\lambda^{i-1}}{(i-1)!}$$

$$= \lambda e^{-\lambda} \cdot e^{\lambda} = \lambda$$

II -> **Var(X) of poisson's Random variable:-**

$$\therefore E[X^2] = \sum_{i=0}^{\infty} i^2 P(i) = \sum_{i=0}^{\infty} i^2 \cdot \frac{e^{-\lambda}\lambda^i}{i!}$$

$$= e^{-\lambda} \cdot \sum_{i=0}^{\infty} \frac{i^2 \lambda^i}{i!} = e^{-\lambda} \cdot \sum_{i=1}^{\infty} \frac{i \lambda^i}{(i-1)!}$$

$$= e^{-\lambda} \cdot \sum_{i=1}^{\infty} \cdot \frac{i \cdot \lambda \cdot \lambda^{i-1}}{(i-1)!} = \lambda e^{-\lambda} \cdot \sum_{i=1}^{\infty} \frac{\lambda^{i-1} \cdot i}{(i-1)!}$$

$\therefore$ Let, $i - 1 = j \Rightarrow i = j + 1$

$$= e^{-\lambda}\lambda \cdot \sum_{j=0}^{\infty} \frac{(j+1) \cdot \lambda^j}{j!}$$

$$= e^{-\lambda}\lambda \cdot \sum_{j=0}^{\infty} \frac{j\lambda^{j}}{j!} + e^{-\lambda}\lambda \cdot \sum_{j=0}^{\infty} \frac{\lambda^{j}}{j!}$$

$$= \lambda \cdot \sum_{j=0}^{\infty} j \cdot \frac{e^{-\lambda}\lambda^{j}}{j!} + \lambda e^{-\lambda} e^{\lambda} = \lambda E[X] + \lambda$$

$$= \lambda \cdot \lambda + \lambda = \lambda^{2} + \lambda$$

$$\therefore Var(x) = E[X^{2}] - (E[X])^{2} = \cancel{\lambda^{2}} + \lambda - \cancel{\lambda^{2}} = \lambda$$

$$\therefore \boxed{Var(x) = E[X] = \lambda}$$

N.B: Poisson's random variable is a limiting case of Binomial Random variable. (IMP)

Supposing that (X) is a Binomial random variable with parameter (n, p) where (n) is the no. of trials & (p) is the probability of success. Let, $\lambda = np. \Rightarrow p = \lambda/n$. Now, the probability mass funct^n of Binomial Random variable is given by: $P(i) = P\{X=i\}$

$$\therefore P(i) = P\{X=i\} = {}^{n}C_{i}\, p^{i}(1-p)^{n-i} \quad \rightarrow ③$$

$$= \frac{n!}{(n-i)!\, i!}\left(\frac{\lambda}{n}\right)^{i}\left(1-\frac{\lambda}{n}\right)^{n-i}$$

$$= \frac{n(n-1)(n-2)\cdots(n-(i-1))}{i!} \cdot \frac{\lambda^{i}}{n^{i}} \cdot \frac{(1-\lambda/n)^{n}}{(1-\lambda/n)^{i}}$$

$$= \frac{\cancel{n^{i}} \cdot \left(1-\frac{1}{n}\right)\left(1-\frac{2}{n}\right)\cdots\left(1-\frac{i-1}{n}\right)}{i! \cdot \cancel{n^{i}}} \cdot \lambda^{i} \cdot \frac{(1-\lambda/n)^{n}}{(1-\lambda/n)^{i}}$$

Now, for large (n), i.e. taking $\boxed{\lim_{n\to\infty}}$ ②, we've:

$$P(i) = \lim_{n\to\infty} \frac{\lambda^{i} \cdot (1-\lambda/n)^{n}}{i!\,(1-\lambda/n)^{i}} = \frac{\lambda^{i}}{i!} \cdot \frac{\lim_{n\to\infty}(1-\lambda/n)^{n}}{\lim_{n\to\infty}(1-\lambda/n)^{i}}$$

$$= \boxed{\lambda^{i} \cdot \frac{e^{-\lambda}}{i!}} \quad ①$$

which is the probability mass funct^n of poisson's random variable.

//→ Continuous Random Variable :-

A random variable whose set of possible values is uncountable is called continuous random variable.

Remark:

ⓐ Probability that a specific event will occur is practically zero.

ⓐ Here, we'll find $P(X)$ falling in a small interval.

Ex: ⓐ the amt. of height of students in a class.

ⓐ the change in temp. throughout a day.

//→ Probability Density Funct^n (PDF) :-

Let (X) be a continuous random variable. If $\exists$ a non-ve funct^n (f) defined $\forall$ real $x \in (-\infty, \infty)$ having the property that for any set (B) of real nos:

$$\boxed{P\{X \in B\} = \int_B f(x)\,dx} \quad \text{---} ①$$

The funct^n (f) is called the PDF of the random variable (X).

Eqn. ① states that the probability that (X) will be in (B) may be obtained by integrating the probability density funct^n over the set (B).

$\because$ (X) must assume some value, so (f) must satisfy:

$$\boxed{P\{X \in (-\infty, \infty)\} = \int_{-\infty}^{\infty} f(x)\,dx = 1}$$

//→ Properties of PDF :-

ⓐ Let, $B = [a, b]$, then: $P\{X \in B\} = \int_a^b f(x)\,dx$

ⓐ If $a = b$, then: $P\{X = a\} = \int_a^a f(x)\,dx = 0$

i.e. the probability that a continuous random variable will assume any fixed value is zero

⊛ $P\{X<a\} = P\{X \leq a\} = P\{X \in (-\infty, a]\}$

$= \int_{-\infty}^{a} f(x)\,dx$ is called CDF of (X).

$$\therefore \boxed{F(a) = P\{X \leq a\} = \int_{-\infty}^{a} f(x)\,dx} \quad \text{(IMP)}$$

where, (F) is the CDF of (X) & (f) is the PDF of (X)

Q: Suppose (X) is a continuous random variable with PDF, given by:

$$f(x) = \begin{cases} c(4x-2x^2), & 0<x<2 \\ 0, & \text{otherwise} \end{cases}$$

⊛ What's the value of c.

⊛ Find $P\{X>1\}$

Ans- ⊛ We know that $\int_{-\infty}^{\infty} f(x)\,dx = 1$

$$= \overset{\nearrow 0}{\cancel{\int_{-\infty}^{0} f(x)\,dx}} + \int_{0}^{2} f(x)\,dx + \overset{\nearrow 0}{\cancel{\int_{2}^{\infty} f(x)\,dx}} = 1$$

$$= \int_{0}^{2} c(4x-2x^2)\,dx = 1$$

$$= 2c\left[2\cdot\frac{x^2}{2} - \frac{x^3}{3}\right]_0^2 = 1$$

$$\Rightarrow 2c\left[x^2 - \frac{x^3}{3}\right]_0^2 = 1 \Rightarrow 2c\left[4 - \frac{8}{3}\right] = 1$$

$$\Rightarrow c = 3/8$$

⊛ $P\{X>1\} = \int_{1}^{\infty} f(x)\,dx = \int_{1}^{2} f(x)\,dx + \overset{\nearrow 0}{\cancel{\int_{2}^{\infty} f(x)\,dx}}$

$$= \int_1^2 \frac{3}{8}(4x-2x^2)\,dx = \frac{6}{8}\left[2\cdot\frac{x^2}{2}-\frac{x^3}{3}\right]_1^2$$

$$= \frac{3}{4}\left[x^2-\frac{x^3}{3}\right]_1^2 = \frac{3}{4}\left[4-\frac{8}{3}-1+\frac{1}{3}\right] = \frac{1}{2} = 0.5$$

Q: The amt. of time in hrs. that a computer funct^{n}s before breaking down is a continuous random variable, with PDF, given by:

$$f(x) = \left.\begin{cases} \lambda e^{-x/100}, & x \geq 0 \\ 0, & x < 0 \end{cases}\right] \rightarrow \text{PDF of ERV}$$

what's the probability that:

ⓐ A computer will funct^n between 50 & 150 hrs. before breaking down.

ⓐ It will funct^n for fewer than 100 hrs.

Ans- ⓐ $\int_{-\infty}^{\infty} f(x)\,dx = 1 = \cancel{\int_{-\infty}^{0} f(x)\,dx}^{\,0} + \int_0^{\infty} f(x)\,dx = 1$

$$= \int_0^{\infty} \lambda e^{-x/100}\,dx = 1$$

$$\Rightarrow \lambda\int_0^{\infty} e^{-x/100}\,dx = 1 \Rightarrow \lambda\left[\frac{e^{-x/100}}{-(1/100)}\right]_0^{\infty} = 1$$

$$\Rightarrow -100\lambda\left[e^{-x/100}\right]_0^{\infty} = 1$$

$$\Rightarrow -100\lambda[0-1] = 1 \Rightarrow 100\lambda = 1$$

$$\Rightarrow \lambda = 1/100$$

$$\therefore P\{50 < x < 150\} = \int_{50}^{150} \frac{e^{-x/100}}{100}\,dx = 0.38$$

ⓐ $P\{X < 100\} = \int_0^{100} \frac{e^{-x/100}}{100}\,dx = 0.632$

II-> Expected value of continuous random variable (X):-

If (X) is a continuous random variable, having PDF $f(x)$, then:

$$\boxed{E[X] = \int_{-\infty}^{\infty} x f(x)\,dx}$$

Q: Find out the E[X] when PDF of (X) is defined by:

$$f(x) = \begin{cases} 2x, & \text{if } 0 \le x \le 1 \\ 0, & \text{otherwise} \end{cases}$$

Ans- We know that, $E[X] = \int_{-\infty}^{\infty} x f(x)\,dx$

$$\therefore \int_{-\infty}^{\infty} x f(x)\,dx = \cancel{\int_{-\infty}^{0} x f(x)\,dx} \xrightarrow{} 0 + \int_{0}^{1} x f(x)\,dx$$

$$+ \cancel{\int_{1}^{\infty} x f(x)\,dx} \xrightarrow{} 0$$

$$\Rightarrow \int_{0}^{1} x f(x)\,dx = \int_{0}^{1} x \cdot 2x\,dx = \int_{0}^{1} 2x^2\,dx$$

$$= \cancel{\int} 2\left[\frac{x^3}{3}\right]_0^1 = 2\left[\frac{1}{3}\right] = \frac{2}{3} = 0.67$$

II-> Relat^n between CDF & PDF of a continuous random variable:-

We know that CDF of a continuous random variable (X) is defined by:

(IMP) $$\boxed{F(a) = P\{X \le a\} = P\{X \in (-\infty, a]\} = \int_{-\infty}^{a} f(x)\,dx}$$

Hence, we can write $\boxed{\frac{d}{da} F(a) = f(a)}$

Q: Let (X) be a continuous random variable with CDF (F_X) & PDF (f_X), then find the PDF of $Y = 2X$.

Ans- $F_Y(a) = P\{Y \le a\} = P\{2X \le a\} = P\{X \le \frac{a}{2}\}$

$$\therefore P\{X \le \frac{a}{2}\} = F_X\left(\frac{a}{2}\right)$$

Now, differentiating both sides, we've:

$$\therefore \frac{d}{da} F_Y(a) = \frac{d}{da} F_X\left(\frac{a}{2}\right)$$

$$\Rightarrow f_Y(a) = f_X\left(\frac{a}{2}\right) \times \frac{1}{2} = \frac{1}{2} f_X\left(\frac{a}{2}\right)$$

Q: The PDF of a continuous random variable (X) is given by:

$$f(x) = \begin{cases} 1, & 0 \le x \le 1 \\ 0, & \text{otherwise} \end{cases}$$

Then find $E[e^X]$

Ans- Let $Y = e^X$ ---- ①

In equn. ①, as (X) varies from zero to 1, so, (Y) varies from 1 to (e).

$\therefore$ For $1 \le t \le e$, $F_Y(t) = P\{Y \le t\}$

$$= P\{e^X \le t\} = P\{\log e^X \le \log t\}$$

$$= P\{X \log e \le \log t\} = P\{X \le \log t\}$$

$$\therefore P\{X \le \log t\} = \int_{-\infty}^{\infty} f(x)\,dx$$

$$= \cancel{\int_{-\infty}^{0} f(x)\,dx}^{\to 0} + \int_{0}^{\log t} f(x)\,dx + \cancel{\int_{\log t}^{\infty} f(x)\,dx}^{\to 0}$$

$$= \int_{0}^{\log t} f(x)\,dx = \int_{0}^{\log t} 1 \cdot dx = [x]_0^{\log t} = \log t - 0$$

$$= \log t$$

Now, differentiating both sides, we've:

$$\frac{d}{dt} F_Y(t) = \frac{d}{dt} \log t$$

$$\Rightarrow f_Y(t) = \frac{1}{t}$$

$$\therefore E[e^X] = E[Y] = \int_1^e t\, f_Y(t)\, dt = \int_1^e \cancel{t} \cdot \frac{1}{\cancel{t}}\, dt$$

$$= \int_1^e 1 \cdot dt = [t]_1^e = e - 1$$

//-> Propositⁿ :-

If (X) is a continuous random variable with PDF $f(x)$, then for any real valued functⁿ (g) of (X), we've:

$$\rightarrow E[g(x)] = \int_{-\infty}^{\infty} g(x) f(x)\, dx$$

$$\therefore E[e^X] = \cancel{\int_{-\infty}^{\infty} e^x\, dx} = \int_{-\infty}^{\infty} e^x f(x)\, dx$$

$$= \overset{\to 0}{\cancel{\int_{-\infty}^{0} f(x)\, dx}} + \int_0^1 e^x f(x)\, dx + \overset{\to 0}{\cancel{\int_1^{\infty} f(x)\, dx}}$$

$$= \int_0^1 (e^x) \cdot 1 \cdot dx = [e^x]_0^1 = e - 1$$

//-> Corollary :-

ⓐ If (a) & (b) are constants, then

$$\boxed{E[aX + b] = aE[X] + b}$$

where, (X) is a continuous random variable.

//-> Variance of continuous random variable:-

$$\therefore Var(X) = E[(X - \mu)^2] = \int_{-\infty}^{\infty} (x - \mu)^2 f(x)\, dx$$

$$= \int_{-\infty}^{\infty} (x^2 + \mu^2 - 2x\mu)\, f(x)\, dx$$

$$= \int_{-\infty}^{\infty} x^2 f(x)\, dx + \mu^2 \int_{-\infty}^{\infty} f(x)\, dx - 2\mu \int_{-\infty}^{\infty} x f(x)\, dx$$

$$\left[\because \int_{-\infty}^{\infty} f(x)\, dx = 1 \text{ for PDF}\right]$$

$$\Rightarrow Var(X) = E[X^2] + \mu^2 - 2\mu \cdot \mu$$

$$= E[X^2] + \mu^2 - 2\mu^2 = E[X^2] - \mu^2$$

$$\therefore \boxed{Var(X) = E[X^2] - (E[X])^2}$$

Q: Find the $Var(X)$, when the PDF of (X) is given by:

$$f(x) = \begin{cases} 2x, & 0 \le x \le 1 \\ 0, & \text{otherwise} \end{cases}$$

Ans:- $\because E[X] = \int_{-\infty}^{\infty} x f(x)\, dx$

$$= \overset{\to 0}{\cancel{\int_{-\infty}^{0} x f(x)\, dx}} + \int_{0}^{1} x f(x)\, dx + \overset{\to 0}{\cancel{\int_{1}^{\infty} x f(x)\, dx}}$$

$$= \int_{0}^{1} x \cdot 2x\, dx = \int_{0}^{1} 2x^2\, dx = \left[\frac{2x^3}{3}\right]_0^1 = \frac{2}{3}$$

$$\therefore E[X^2] = \int_{-\infty}^{\infty} x^2 f(x)\, dx$$

$$= \overset{\to 0}{\cancel{\int_{-\infty}^{0} x^2 f(x)\, dx}} + \int_{0}^{1} x^2 f(x)\, dx + \overset{\to 0}{\cancel{\int_{1}^{\infty} x^2 f(x)\, dx}}$$

$$= \int_{0}^{1} x^2 \cdot 2x\, dx = \int_{0}^{1} 2x^3\, dx = \left[\frac{2x^4}{4}\right]_0^1 = \frac{2}{4} = \frac{1}{2}$$

$$\therefore Var(X) = E[X^2] - (E[X])^2$$

$$= \frac{1}{2} - \frac{4}{9} = \frac{1}{18}$$

N.B.: $\boxed{Var(aX+b) = a^2 Var(X)}$, where (X) is a continuous random variable.

II → Uniform Random Variable :-

(X) is said to be a Uniform Random Variable on the interval (α, β) if the PDF of (X) is given by:

$$\boxed{f(x) = \begin{cases} \frac{1}{\beta - \alpha}, & \alpha < x < \beta \\ 0, & \text{otherwise} \end{cases}}$$ PDF of URV (X)

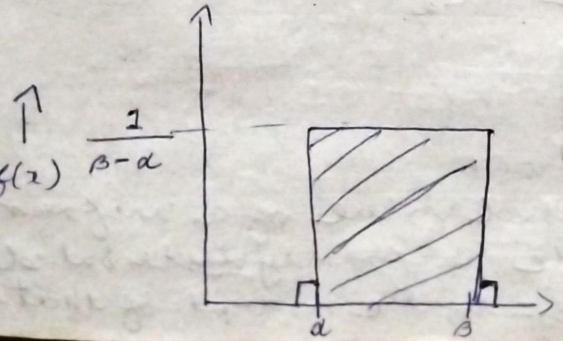
$$\therefore \int_{\alpha}^{\beta} f(x)dx = \int_{\alpha}^{\beta} \frac{1}{\beta-\alpha} dx = \frac{1}{(\beta-\alpha)} \cdot (\beta - \alpha) = 1$$

Clearly, $f(x)$ is a PDF.

Now, the CDF of uniform random variable on the interval (α, β) is given by:

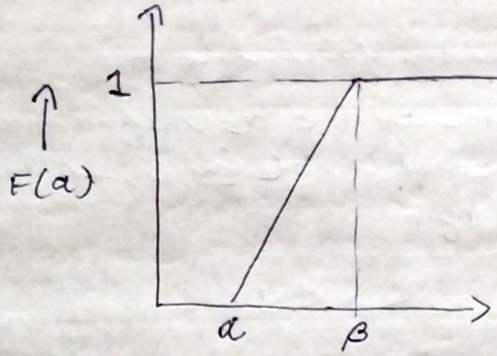
$$\boxed{F(a) = \begin{cases} 0, & a \le \alpha \\ \frac{a-\alpha}{\beta-\alpha}, & \alpha < a < \beta \\ 1, & a > \beta \end{cases}}$$ CDF of URV(X)

Graphically, the PDF of uniform random variable is portrayed as the rectangle, where $(\beta - \alpha)$ is the base, & $\frac{1}{\beta-\alpha}$ is the height.



[As the dist. between (α) & (β) increases the density at any particular value within the distribut^n boundary decreases]

[$\because$ the PDF integrates to 1, the height of PDF decreases as the base length increases]



//-> **Remark:-**

Ⓐ In particular, when $(\alpha, \beta) = (0, 1)$, then:

$$f(x) = \begin{cases} 1, & 0 < x < 1 \\ 0, & \text{otherwise} \end{cases}$$

⊛ For $a \le c < x \le d \le b$, we've:

$$P\{c \le X \le d\} = \int_c^d \frac{1}{b-a}\,dt = \frac{d-c}{b-a}$$

//-> **Expectat^n of Uniform Random variable:-**

$$E[X] = \int_{-\infty}^{\infty} x f(x)\,dx = \int_{\alpha}^{\beta} \frac{x}{\beta-\alpha}\,dx$$

$$= \frac{1}{\beta-\alpha}\left[\frac{x^2}{2}\right]_{\alpha}^{\beta} = \frac{\beta^2-\alpha^2}{2(\beta-\alpha)} = \frac{(\beta+\alpha)\cancel{(\beta-\alpha)}}{2\cancel{(\beta-\alpha)}}$$

✓ $$\Rightarrow \boxed{E[X] = \frac{\beta+\alpha}{2}}$$

i.e. the expected value of a uniform random variable, uniformly distributed over some interval is = to the mid-pt. of that interval

11→ Variance of Uniform Random Variable:-

$$\therefore E[X^2] = \int_{\alpha}^{\beta} x^2 f(x)\,dx = \int_{\alpha}^{\beta} \frac{x^2}{\beta-\alpha}\,dx$$

$$= \frac{1}{\beta-\alpha}\left[\frac{x^3}{3}\right]_{\alpha}^{\beta} = \frac{\beta^3-\alpha^3}{3(\beta-\alpha)}$$

$$= \frac{\cancel{(\beta-\alpha)}(\beta^2+\alpha\beta+\alpha^2)}{3\cancel{(\beta-\alpha)}} = \frac{\beta^2+\alpha\beta+\alpha^2}{3}$$

$$\therefore Var(X) = E[X^2] - (E[X])^2$$

$$= \frac{\beta^2+\alpha\beta+\alpha^2}{3} - \frac{\beta^2+\alpha^2+2\alpha\beta}{4}$$

$$= \frac{\beta^2+\alpha\beta+\alpha\beta-\alpha\beta+\alpha^2}{3} - \frac{\beta^2+\alpha^2+2\alpha\beta}{4}$$

$$= \frac{(\beta+\alpha)^2-\alpha\beta}{3} - \frac{(\beta+\alpha)^2}{4}$$

$$= \frac{4(\beta+\alpha)^2-4\alpha\beta-3(\beta+\alpha)^2}{12} = \frac{(\beta+\alpha)^2-4\alpha\beta}{12}$$

$$= \frac{\beta^2+\alpha^2+2\alpha\beta-4\alpha\beta}{12} = \frac{\beta^2+\alpha^2-2\alpha\beta}{12}$$

$$\Rightarrow \boxed{Var(X) = \frac{(\beta-\alpha)^2}{12}}$$

Q: If (X) is uniformly distributed over the interval (0, 10), then calculate the probability that:

ⓐ $X<3$ ⓑ $X>6$ ⓒ $3<X<8$

Ans- Here, $\alpha = 0$ & $\beta = 10$

$$\therefore ⓐ\ P\{X<3\} = \int_0^3 \frac{1}{10-0}\,dx = \int_0^3 \frac{1}{10}\,dx$$

$$= \frac{3-0}{10} = \frac{3}{10}$$

ⓐ $P\{x>6\} = \int_{6}^{10} \frac{1}{10-0}\, dx = \frac{10-6}{10-0} = \frac{4}{10} = \frac{2}{5}$

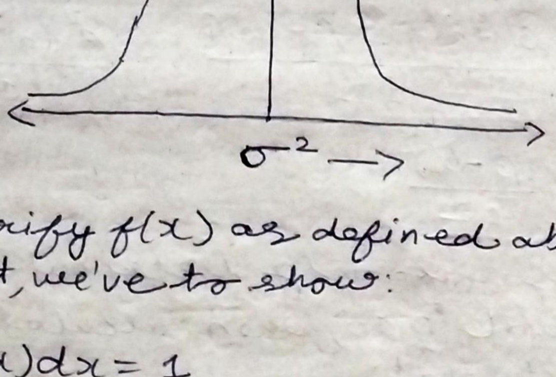
ⓐ $P\{3<X<8\} = \int_{3}^{8} \frac{1}{10-0}\, dx = \frac{8-3}{10-0} = \frac{5}{10} = \frac{1}{2}$

II-> Normal random variable (Gaussian or Laplace Random variable):-

(X) is said to be the Normal random variable or simply that (X) is normally distributed with parameters (μ) & (σ^2) if the PDF of (X) is given by:

$$\boxed{f(x) = \frac{1}{\sigma\sqrt{2\pi}} \cdot e^{-\frac{(x-\mu)^2}{2\sigma^2}}, \quad x \in (-\infty, \infty)}$$

This density funct^n is a bell shaped curve i.e. symmetric about (μ)



To verify f(x) as defined above is a PDF, for that, we've to show:

$$\int_{-\infty}^{\infty} f(x)\,dx = 1$$

$$\Rightarrow \frac{1}{\sigma\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{-\frac{(x-\mu)^2}{2\sigma^2}}\, dx = 1$$

$\therefore$ Let, $y = \frac{x-\mu}{\sigma} \Rightarrow dy = \frac{dx}{\sigma}$

$$\therefore \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{-\frac{(x-\mu)^2}{2\sigma^2}} \frac{dx}{\sigma} = 1$$

$$\Rightarrow \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{-y^2/2}\, dy = 1$$

Now, let $I = \int_{-\infty}^{\infty} e^{-y^2/2}\, dy \quad \ldots ①$

also, $I = \int_{-\infty}^{\infty} e^{-x^2/2}\, dx$, where (x) is a dummy variable. $\ldots$ ⓘⓘ

Now, multiplying equn ① & ⓘⓘ with each other, we've:

$$I^2 = \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} e^{-y^2/2} \cdot e^{-x^2/2}\, dy\, dx$$

Again, let, $x = r\cos\theta$ & $y = r\sin\theta$

$$\therefore I^2 = \int_{0}^{\infty} \int_{0}^{2\pi} e^{-(x^2+y^2)/2}\, r\, dr\, d\theta$$

$$= \int_{0}^{\infty} \int_{0}^{2\pi} e^{-r^2/2}\, r\, dr\, d\theta$$

$(\because x^2+y^2 = r^2\cos^2\theta + r^2\sin^2\theta = r^2(\cos^2\theta + \sin^2\theta)$
$= r^2)$

So, when, (x) & (y) both varies from $-\infty$ to ∞, (r) varies from zero to ∞ & (θ) varies from zero to 2π.

$$\therefore I^2 = 2\pi \int_{0}^{\infty} r e^{-r^2/2}\, dr$$

Let, $r^2 = t \Rightarrow 2r\,dr = dt \Rightarrow r\,dr = dt/2$

$$\therefore I^2 = 2\pi \int_0^{\infty} e^{-t/2} \frac{dt}{2} = \pi \int_0^{\infty} e^{-t/2}\, dt = \pi\left[-2e^{-t/2}\right]_0^{\infty}$$

$\Rightarrow -2\pi(0-1) = 2\pi$

$\therefore I^2 = 2\pi \Rightarrow I = \sqrt{2\pi}$

So, $\frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{-y^2/2} dy = 1$

$\Rightarrow \frac{1}{\sqrt{2\pi}} \times I = 1 \quad \Rightarrow \frac{1}{\cancel{\sqrt{2\pi}}} \times \cancel{\sqrt{2\pi}} = 1$

$\Rightarrow \boxed{1 = 1}$ (HP)

N.B: If (X) is normally distributed with parameter (μ) & (σ^2) i.e.

$$\boxed{X : (\mu, \sigma^2)}$$

then (Y) i.e. $Y = aX + b$ is normally distributed with parameter:

$$\boxed{Y : (a\mu + b, a^2\sigma^2)}$$

We know, $f_X = \frac{1}{\sigma\sqrt{2\pi}} \cdot e^{-\frac{(x-\mu)^2}{2\sigma^2}}$

Proof: $F_Y(x) = P\{Y \le x\} = P\{aX + b \le x\}$

$= P\{aX \le x - b\} = P\{X \le \frac{x-b}{a}\}$

$= F_X\left(\frac{x-b}{a}\right)$

$\therefore$ Differentiating both sides, we've:

$\frac{d}{dx} F_Y(x) = \frac{d}{dx} F_X\left(\frac{x-b}{a}\right)$

$\Rightarrow f_Y(x) = f_X\left(\frac{x-b}{a}\right) \cdot \frac{1}{a}$

$= \frac{1}{a} \cdot \frac{1}{\sigma\sqrt{2\pi}} \cdot e^{-\frac{\left[\left(\frac{x-b}{a}\right) - \mu\right]^2}{2\sigma^2}}$

$= \frac{1}{a\sigma\sqrt{2\pi}} \cdot e^{-\frac{\left[\left(\frac{x-b}{a}\right) - \mu\right]^2}{2\sigma^2}}$

which shows that (Y) is a normal random

$\boxed{Y : (a\mu + b, a^2\sigma^2)}$ where, $\boxed{Y = aX + b}$

//→ Expectatⁿ of normal random variable:-

Let (X) be a normal random variable

$$\therefore E[X] = \int_{-\infty}^{\infty} x f(x)\, dx$$

$$= \int_{-\infty}^{\infty} \frac{x}{\sigma\sqrt{2\pi}} \cdot e^{-\frac{(x-\mu)^2}{2\sigma^2}}\, dx$$

$$= \frac{1}{\sigma\sqrt{2\pi}} \int_{-\infty}^{\infty} x \cdot e^{-\frac{(x-\mu)^2}{2\sigma^2}}\, dx$$

Let, $\frac{x-\mu}{\sigma} = z \implies x = \sigma z + \mu$

$\implies dx = \sigma dz \implies \frac{dx}{\sigma} = dz$

$$\therefore \frac{1}{\sigma\sqrt{2\pi}} \int_{-\infty}^{\infty} (\sigma z + \mu) \cdot e^{-z^2/2}\, dx$$

$$= \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} (\sigma z + \mu) \cdot e^{-z^2/2}\, dz$$

$$= \frac{1}{\sqrt{2\pi}} \cdot \sigma \int_{-\infty}^{\infty} z e^{-z^2/2}\, dz + \frac{\mu}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{-z^2/2}\, dz$$

But $\int_{-\infty}^{\infty} z e^{-z^2/2}\, dz = 0$, since the function $z e^{-z^2/2}$ is an odd functⁿ. So; we've:

$$0 + \frac{\mu}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{-z^2/2}\, dz$$

$$= \mu \cdot \cancelto{1}{\frac{1}{\sigma\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{-\frac{(x-\mu)^2}{2\sigma^2}}\, dx} = \mu \times 1 = \mu$$

$$\therefore \boxed{E[X] = \mu}$$

II-> Variance of a normal random variable:-

$$E[X^2] = \int_{-\infty}^{\infty} x^2 f(x)\,dx$$

$$= \frac{1}{\sigma\sqrt{2\pi}} \int_{-\infty}^{\infty} x^2 e^{-\frac{(x-\mu)^2}{2\sigma^2}} \cdot dx$$

Let, $\frac{x-\mu}{\sigma} = z \Rightarrow x = \sigma z + \mu$

$\Rightarrow dx = \sigma dz \Rightarrow \frac{dx}{\sigma} = dz$

$$\therefore \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} (\sigma z + \mu)^2 \cdot e^{-z^2/2}\,dz$$

$$= \frac{1}{\sqrt{2\pi}} \int (\sigma^2 z^2 + 2\sigma z \mu + \mu^2) \cdot e^{-z^2/2}\,dz$$

$$= \frac{\sigma^2}{\sqrt{2\pi}} \int_{-\infty}^{\infty} z^2 e^{-z^2/2}\,dz + \cancelto{0}{\frac{2\sigma\mu}{\sqrt{2\pi}} \int_{-\infty}^{\infty} z e^{-z^2/2}\,dz}$$

$$+ \mu^2 \cancelto{1}{\int_{-\infty}^{\infty} \frac{e^{-z^2/2}}{\sqrt{2\pi}}}\,dz$$

$$= \frac{\sigma^2}{\sqrt{2\pi}} \int_{-\infty}^{\infty} z^2 e^{-z^2/2}\,dz + \mu^2 \quad \text{----} \quad ①$$

Now, let $I_1 = \int_{-\infty}^{\infty} z^2 e^{-z^2/2}\,dz$

$$= \int_{-\infty}^{\infty} z \cdot z e^{-z^2/2}\,dz$$

$$= z \int z e^{-z^2/2}\,dz - \int \left(1 \cdot z e^{-z^2/2}\,dz\right) dz$$

Again, let $z^2/2 = t \Rightarrow z\,dz = dt$

$$\therefore z \int e^{-t}\,dt - \int \left[e^{-t}\,dt \right] dz$$

$= z(-e^{-t}) - \int (-e^{-t})\, dz$

$= z(-e^{-t}) + \int e^{-t}\, dz$

$= z\left[-e^{-z^2/2}\right]_{-\infty}^{\infty} + \int_{-\infty}^{\infty} e^{-z^2/2}\, dz$

$= 0 + \int_{-\infty}^{\infty} e^{-z^2/2}\, dz$

$\therefore E[x^2] = \frac{\sigma^2}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{-z^2/2}\, dz + \mu^2$ (from ①)

$= \frac{\sigma^2}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{-\frac{(x-\mu)^2}{2\sigma^2}} \frac{dx}{\sigma} + \mu^2$

$= \sigma^2 \left[\cancelto{1}{\frac{1}{\sigma\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{-\frac{(x-\mu)^2}{2\sigma^2}}\, dx}\right] + \mu^2$

$= (\sigma^2 \times 1) + \mu^2 = \sigma^2 + \mu^2 \quad \therefore \boxed{E[x^2] = \sigma^2 + \mu^2}$

$\therefore Var(X) = E[x^2] - (E[x])^2$

$= \sigma^2 + \mu^2 - (\mu)^2 = \sigma^2 + \cancel{\mu^2} - \cancel{\mu^2} = \sigma^2$

$\therefore \boxed{E[x^2] = Var(X) + \mu^2}$

$\therefore \boxed{Var(X) = \sigma^2}$

N.B: The random variable $\boxed{(Z) = \frac{X-\mu}{\sigma}}$ is normally distributed with parameter $\boxed{\mu = 0 \,\&\, \sigma = 1}$, Such ~~that~~ a normal random variable is called std. or unit normal random variable.

II → Expectat^n & Variance of std. normal random variable:-

ⓐ $E[Z] = E\left[\frac{X-\mu}{\sigma}\right] = E\left[\frac{X}{\sigma} - \frac{\mu}{\sigma}\right] = \frac{1}{\sigma} E[X] - \frac{\mu}{\sigma}$

$= \cancel{\frac{\mu}{\sigma}} - \cancel{\frac{\mu}{\sigma}} = 0 \quad \therefore \boxed{E[Z] = 0}$

ⓑ $Var(Z) = Var\left(\frac{X-\mu}{\sigma}\right) = Var\left(\frac{X}{\sigma} - \frac{\mu}{\sigma}\right)$

$= \frac{1}{\sigma^2} Var(X) = \frac{1}{\cancel{\sigma^2}} \times \cancel{\sigma^2} = 1 \quad \therefore \boxed{Var(Z) = 1}$

II → CDF of std. normal random variable:-

The CDF of std. normal random variable is given by:

$$\phi(x) = P\{Z \le x\} = \frac{1}{\sigma\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{-\frac{(z-\mu)^2}{2\sigma^2}} dz$$

$$= \frac{1}{\sigma\sqrt{2\pi}} \left[\int_{-\infty}^{x} e^{-\frac{(z-\mu)^2}{2\sigma^2}} dz + \cancel{\int_{x}^{\infty} e^{-\frac{(z-\mu)^2}{2\sigma^2}} dz} \to 0 \right]$$

$$= \frac{1}{\sigma\sqrt{2\pi}} \int_{-\infty}^{x} e^{-\frac{(z-\mu)^2}{2\sigma^2}} dz$$

But for std. normal random variable, $\mu = 0$ & $\sigma = 1$

$$\therefore \boxed{\phi(x) = P\{Z \le x\} = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{x} e^{-z^2/2} dz}$$

N.B: $\boxed{\phi(-x) = 1 - \phi(x)}$, where (ϕ) is the CDF of the std. normal random variable.

Q: If (X) is a normal random variable with parameter $\mu = 3$ & $\sigma^2 = 9$, then find:

ⓐ $P\{2 < X < 5\}$

ⓑ $P\{X > 0\}$ ⓒ $P\{|X-3| > 6\}$

Ans- ⓐ $P\{2 < X < 5\} = P\left\{\frac{2-\mu}{\sigma} < \frac{X-\mu}{\sigma} < \frac{5-\mu}{\sigma}\right\}$

$= P\left\{\frac{2-3}{3} < \frac{X-3}{3} < \frac{5-3}{3}\right\} = P\left\{-\frac{1}{3} < Z < \frac{2}{3}\right\}$

$\left(\because Z = \frac{X-\mu}{\sigma}\right)$

Same logical Meaning

$\Rightarrow P\left\{Z < \frac{2}{3}\right\} - P\left\{Z < -\frac{1}{3}\right\} = \phi\left(\frac{2}{3}\right) - \phi\left(-\frac{1}{3}\right)$

$$= \phi\left(\frac{2}{3}\right) - 1 + \phi\left(\frac{1}{3}\right) = 0.7454 + 0.6293 - 1$$
$$= 0.3747$$

ⓒ $P\{X>0\} = P\left\{\frac{X-\mu}{\sigma} > \frac{0-\mu}{\sigma}\right\} = P\left\{\frac{X-3}{3} > \frac{0-3}{3}\right\}$

$$= P\{Z > -1\} = 1 - P\{Z \le -1\} = 1 - \phi(-1)$$
$$= 1 - (1-\phi(1)) = \cancel{1} - \cancel{1} + \phi(1) = \phi(1) = 0.8413$$

ⓓ $P\{|X-3| > 6\} = P\{-6 > X-3 > 6\}$

$$= P\left\{\frac{-6}{3} > \frac{X-3}{3} > \frac{6}{3}\right\} = P\{-2 > Z > 2\}$$
$$= P\{Z \le -2\} + P\{Z > 2\}$$
$$= P\{Z \le -2\} + 1 - P\{Z \le 2\}$$
$$= \phi(-2) + 1 - \phi(2) = 1 - \phi(2) + 1 - \phi(2)$$
$$= 2(1-\phi(2)) = 2(1-0.9722) = 0.0476$$

11 -> Exponential random variable:-

A continuous random variable, whose PDF is given by:

$$\boxed{f(x) = \begin{cases} \lambda e^{-\lambda x}, & x \ge 0 \\ 0, & x < 0 \end{cases}}$$ PDF of ERV(X)

for some $\lambda > 0$, is said to be an exponential random variable with parameter (λ).

Now, the CDF of exponential random variable is given by:

$$F(a) = P\{X \le a\} = \int_{-\infty}^{\infty} f(x)\,dx$$

$$= \cancel{\int_{-\infty}^{0} f(x)\,dx}^{\to 0} + \int_{0}^{a} f(x)\,dx + \cancel{\int_{a}^{\infty} f(x)\,dx}^{\to 0}$$

$$= \int_{0}^{a} \lambda e^{-\lambda x}\,dx = \frac{\lambda}{-\lambda}\left[e^{-\lambda x}\right]_0^a = -\left[e^{-\lambda a} - 1\right]$$

$= 1 - e^{-\lambda a}$ $\therefore$ $\boxed{F(a) = P\{X \le a\} = 1 - e^{-\lambda a}}$ CDF of ERV(X)

11 → Mean & Variance of Exponential random variable: –

$$\therefore E[X] = \int_{-\infty}^{\infty} x f(x)\,dx = \int_{0}^{\infty} x\cdot \lambda e^{-\lambda x}\,dx$$

$$= \lambda \int_{0}^{\infty} x e^{-\lambda x}\,dx$$

$$= \lambda\left[x\cdot \frac{e^{-\lambda x}}{-\lambda} - 1\cdot \frac{e^{-\lambda x}}{\lambda^2}\right]_0^{\infty}$$

$$= \lambda\left[(0-0) - \left(0 - \frac{1}{\lambda^2}\right)\right] = \lambda \times \frac{1}{\lambda^2} = \frac{1}{\lambda}$$

$$\therefore \boxed{E[X] = \frac{1}{\lambda}}$$

Now, $Var(X) = E[X^2] - (E[X])^2$.

$$\therefore E[X^2] = \int_{-\infty}^{\infty} x^2 f(x)\,dx = \int_{0}^{\infty} x^2\cdot \lambda e^{-\lambda x}\,dx$$

$$= \lambda \int_{0}^{\infty} x^2 e^{-\lambda x}\,dx$$

$$= \lambda\left[x^2\cdot \frac{e^{-\lambda x}}{-\lambda} - 2x\cdot \frac{e^{-\lambda x}}{\lambda^2} + 2\cdot \frac{e^{-\lambda x}}{-\lambda^3}\right]_0^{\infty}$$

$$= \lambda\left[(0-0) - (0-0) + 2\left(0 - \frac{1}{-\lambda^3}\right)\right]$$

$$= \lambda\left[\frac{2}{\lambda^3}\right] = \frac{2}{\lambda^2}$$

$$\therefore Var(X) = E[X^2] - (E[X])^2 = \frac{2}{\lambda^2} - \frac{1}{\lambda^2} = \frac{1}{\lambda^2}$$

$$\therefore \boxed{Var(X) = \frac{1}{\lambda^2}} \Rightarrow \boxed{Var(X) = (E[X])^2}$$

Thus the mean of the exponential random variable is the reciprocal of its parameter (λ).

| the variance is the square of its mean.

Q: suppose that the length of a phone call in minutes is an exponential random variable, with parameter $\lambda = 1/10$. If someone arrives immediately ahead of you at a public telephone booth, find the probability that ~~you~~ he will've to wait:

ⓐ More than 10 minutes.

ⓑ Between 10 to 20 minutes.

Ans- Let X: length of a phone call in minutes.

ⓐ $$P\{X>10\} = \int_{10}^{\infty} \lambda e^{-\lambda x}\,dx = \int_{10}^{\infty} 0.1\, e^{-0.1x}\,dx$$

$$= 0.1\left[\frac{e^{-0.1x}}{-0.1}\right]_{10}^{\infty} = 0.1\left[0 - \frac{e^{-1}}{-0.1}\right]$$

$$= e^{-1}$$

-OR-

$$P\{X>10\} = 1 - P\{X \le 10\} = 1 - F(10)$$

$$= 1 - \left[1 - e^{-0.1\times 10}\right] = e^{-1} = 0.3679\ (36.79\%)$$

ⓑ $$P\{10 < X < 20\} = P\{X < 20\} - P\{X < 10\}$$

$$= F(20) - F(10) = \left[1 - e^{-0.1\times 20}\right] - \left[1 - e^{-0.1\times 10}\right]$$

$$= e^{-1} - e^{-2} = 0.2325\ (23.25\%)$$

II-> Hypergeometric distribut^n :-

[It's a discrete probability distribut^n that describes probability of (k) success in (n) draws without replacement from a finite populat^n of size (N) that ~~coincides~~ contains exactly (K) objects (objects with success feature), where in each draw is either a success or a failure.

[Here, probability of success changes from trial-to-trial.

∴ Probability mass funct^n (PMF) of hypergeometric distribut^n (HGD) is given by:

$$\boxed{P(k) = P\{X = k\} = \frac{\binom{K}{k}\binom{N-K}{n-k}}{\binom{N}{n}}}$$

Q: Suppose in a box, there are 4 white & 6 black balls. If choosing a black ball is considered as a success, then what's the probability of 3 successes if we draw a ball from a box 5 times.

Ans- Let N = Populat^n size = Total no. of objects = 10

K = Objects with success feature = 6

n = No. of trials = 5

k = No. of successes = 3

$$\therefore P(3) = P\{X=3\} = \frac{\binom{6}{3}\binom{10-6}{5-3}}{\binom{10}{5}} = \frac{10}{21}$$

$$= 0.476$$

Q: 6 doctors & 19 nurses attend a concert or small conference. All 25 names are put in a hat & 5 names are randomly picked without replacement. what's the probability that 4 doctors & 1 nurse are picked.

Ans- Let N = 25

K = 6

n = 5

k = 4

$$\therefore P(4) = P\{X=4\} = \frac{\binom{6}{4}\binom{25-6}{5-4}}{\binom{25}{5}} = \frac{\binom{6}{4}\binom{19}{1}}{\binom{25}{5}}$$

$$= \frac{19}{3542} = 0.00536$$

// -> Mean of hypergeometric distribut^n :-

we know, $E[X] = \sum_{k=0}^{n} kP(k)$

$$= \sum_{k=0}^{n} k \cdot \frac{\binom{K}{k}\binom{N-K}{n-k}}{\binom{N}{n}} = \sum_{k=1}^{n} k \cdot \frac{\frac{K}{k}\binom{K-1}{k-1}\binom{N-K}{n-k}}{\frac{N}{n}\binom{N-1}{n-1}}$$

$$\left[\because {}^{n}C_{r} = \frac{n}{r}\,{}^{(n-1)}C_{(r-1)}\right]$$

$$= \frac{nK}{N} \cdot \sum_{k=1}^{n} \frac{\binom{K-1}{k-1}\binom{N-K}{n-k}}{\binom{N-1}{n-1}}$$

Let, $k-1=m \Rightarrow k=m-1$

when, $k=1 \Rightarrow m=0$

$$\therefore \frac{nK}{N}\sum_{m=0}^{n-1}\frac{\binom{K-1}{m}\binom{\overset{(N-1)-(K-1)}{N-K}}{(n-1)-m}}{\binom{N-1}{n-1}} = \frac{nK}{N}\times 1 = \frac{nK}{N}$$

$$\frac{\binom{K-1}{m}\binom{(N-1)-(K-1)}{(n-1)-m}}{\binom{N-1}{n-1}}$$

$$\therefore \boxed{E[X]=\frac{nK}{N}}$$

II → Variance of hypergeometric distributⁿ:-

$$Var(X) = E[X^2]-(E[X])^2$$

$$= E[X(X-1)+X]-(E[X])^2$$

$$= E[X(X-1)]+E[X]-(E[X])^2$$

$$\therefore E[X(X-1)] = \sum_{k=0}^{n} k(k-1)P(k)$$

$$= \sum_{k=0}^{n} k(k-1)\frac{\binom{K}{k}\binom{N-K}{n-k}}{\binom{N}{n}}$$

$$= \sum_{k=2}^{n} k(k-1)\frac{\frac{K(K-1)}{k(k-1)}\binom{K-2}{k-2}\binom{N-K}{n-k}}{\frac{N(N-1)}{n(n-1)}\binom{N-2}{n-2}}$$

$$= \frac{n(n-1)}{N(N-1)}K(K-1)\cdot\sum_{k=2}^{n}\frac{\binom{K-2}{k-2}\binom{N-K}{n-k}}{\binom{N-2}{n-2}}$$

Let, $k-2=m \Rightarrow k=m+2$

when, $k=2 \Rightarrow m=0$

$$= \frac{n(n-1)K(K-1)}{N(N-1)}\cdot\sum_{m=0}^{n-2}\frac{\binom{K-2}{m}\binom{\overset{(N-2)-(K-2)}{N-K}}{(n-2)-m}}{\binom{N-2}{n-2}}$$

$$= \frac{n(n-1)K(K-1)}{N(N-1)}\times 1 = \frac{n(n-1)K(K-1)}{N(N-1)}$$

$$\therefore Var(X) = E[X(X-1)]+E[X]-(E[X])^2$$

$$= \frac{n(n-1)K(K-1)}{N(N-1)}+\frac{nK}{N}-\frac{n^2K^2}{N}$$

$$\Rightarrow \boxed{Var(x) = \frac{nK}{N}\left(\frac{N-K}{N}\right)\left(\frac{N-n}{N-1}\right)}$$

$$\Rightarrow \boxed{Var(x) = E[x]\left(\frac{N-K}{N}\right)\left(\frac{N-n}{N-1}\right)}$$

N.B: Hypergeometric distributⁿ tends to binomial distributⁿ as $N \to \infty$.

$\therefore \frac{K}{N} \to p$ (Probability of success)

$n \to \infty$ BD → PD $N \to \infty$ HD → BD

$$\Rightarrow E[X] = \frac{nK}{N} = np$$

$$\Rightarrow Var(x) = \frac{nK}{N}\left(\frac{N-K}{N}\right)\cancel{\left(\frac{N-n}{N-1}\right)}^{\to 1} = np(1-p)$$

II → Bivariate Distributⁿ (Jointly distributed random variable):-

⇒ Continuous Case:

I → Joint CDF of Bivariate random variable (X) & (Y):-

For 2 random variables (X) & (Y), the joint CDF of (X) & (Y) is given by:

$$\boxed{F_{XY}(a,b) = P\{X \le a, Y \le b\}}$$

where, $-\infty < a, b < \infty$

The distributⁿ of (X) & (Y) can be obtained from the joint CDF of (X) & (Y) as ~~above~~ follows, & are termed as marginal CDF of (X) & (Y) respectively.

I → Marginal CDF of (X):-

$$F_X(a) = P\{X \le a\} = P\{X \le a, Y < \infty\}$$

$$= P\left(\lim_{b\to\infty}\{X \le a, Y \le b\}\right)$$

$$= \lim_{b\to\infty} P\{X \le a, Y \le b\} = \lim_{b\to\infty} F_{XY}(a,b)$$

$$= F_{XY}(a,\infty) \qquad \therefore \boxed{F_X(a) = F_{XY}(a,\infty)}$$

$$\boxed{F_Y(b) = F_{XY}(\infty, b)}$$

⇒ Discrete Case:

1-> Joint PMF of Bivariate random variable (X) & (Y):-

Suppose (X) & (Y) are both discrete random variables. Thus, the joint PMF of (X) & (Y) is given by:

$$\boxed{P_{XY}(a, b) = P\{X = a, Y = b\}}$$

Now, we can define the joint range for (X) & (X) as:

$$\boxed{R_{XY} = \{(x, y) \mid x \in R_X \ \& \ y \in R_Y, P_{XY}(x, y) \geq 0\}}$$

Here, R_{XY} is a subset of the cross product of R_X & R_Y. Hence, for 2 discrete random variables (X) & (Y), we've:

$$\boxed{\sum_{(x,y) \in R_{XY}} P_{XY}(x, y) = 1}$$

1-> Marginal PMF of (X):-

The joint PMF contains all the informatⁿ regarding the distributⁿ of (X) & (Y). So, we can obtain the PMF of (X) from its joint PMF with (Y).

$$\therefore P_X(x) = P\{X = x\} = \sum_{y_j \in R_Y} P\{X = x, Y = y_j\}$$

$$= \sum_{y_j \in R_Y} P_{XY}(x, y_j)$$

$$\therefore \boxed{P_X(x) = \sum_{y_j \in R_Y} P_{XY}(x, y_j)}$$

1-> Marginal PMF of (Y):-

$$\boxed{P_Y(y) = \sum_{x_i \in R_X} P_{XY}(x_i, y)}$$

Q: Consider 2 random variables (X) & (Y), with joint PMF, given by:-

$X\downarrow$ \ Y	$Y=0$	$Y=1$	$Y=2$
$X=0$	$1/6$	$1/4$	$1/8$
$X=1$	$1/8$	$1/6$	$1/6$

Find, ⓐ $P\{X=0, Y\leq 1\}$

ⓑ Find the marginal PMF of X & Y.

Ans- ⓐ $P\{X=0, Y\leq 1\}$

$= P\{X=0, Y=0\} + P\{X=0, Y=1\}$

$= \frac{1}{6} + \frac{1}{4} = \frac{5}{12}$

ⓑ $P_X(x) = \sum_{y_j=0}^{2} P_{XY}(x, y_j)$

$\therefore P_X(0) = \frac{1}{6} + \frac{1}{4} + \frac{1}{8} = \frac{13}{24}$

$\therefore P_X(1) = \frac{1}{8} + \frac{1}{6} + \frac{1}{6} = \frac{11}{24}$

$P_Y(y) = \sum_{x_i=0}^{1} P_{XY}(x_i, y)$

$\therefore P_Y(0) = \frac{1}{6} + \frac{1}{8} = \frac{7}{24}$

$\therefore P_Y(1) = \frac{1}{4} + \frac{1}{6} = \frac{5}{12}$

$\therefore P_Y(2) = \frac{1}{8} + \frac{1}{6} = \frac{7}{24}$

=> Continuous Case:

1-> Joint PDF of (X) & (Y) i.e. Bivariate random variable:-

Let (X) & (Y) are 2 ~~contin~~ continuous random variables. Then, ∃ a funct^n $f(x,y)$, defined ∀ (x,y), having the property that for every set (C), i.e. of pairs of real nos:

$$\boxed{P\{(x,y) \in C\} = \iint f(x,y)\,dx\,dy}$$

The funct^n $f(x,y)$ is called the joint PDF of (X) & (Y). Here, $f(x,y) \geq 0$

✓

$$\therefore \boxed{\iint_{R_{XY}} f(x,y)\,dx\,dy = 1}$$

If (A) & (B) are any sets of real nos, & $C = \{(x,y) \mid x \in A, y \in B\}$, then:

✓

$$\boxed{P\{X \in A, Y \in B\} = \int_B \int_A f(x,y)\,dx\,dy}$$

→ Joint CDF of Bivariate random variable (X) & (Y) :-

Here, it's given by as follows:

✓

$$\boxed{F_{XY}(a,b) = P\{X \leq a, Y \leq b\} = \int_{-\infty}^{b}\int_{-\infty}^{a} f(x,y)\,dx\,dy}$$

→ Marginal PDF of (X) :-

$$F_X(a) = P\{X \leq a, Y < \infty\} = \int_{-\infty}^{\infty}\int_{-\infty}^{a} f(x,y)\,dx\,dy$$

$$\Rightarrow \int_{-\infty}^{a}\left(\int_{-\infty}^{\infty} f(x,y)\,dy\right)dx = \int_{-\infty}^{a} f_X(x)\,dx$$

Now, differentiating both sides of the above eqn. w.r.t (x), we've:

$$\int_{-\infty}^{\infty} f(x,y)\,dy = f_X(x)$$

✓

$$\therefore \boxed{f_X(x) = \int_{-\infty}^{\infty} f(x,y)\,dy}$$

→ Marginal PDF of (Y) :-

$$F_Y(b) = P\{X \le \infty, Y \le b\} = \int_{-\infty}^{b}\int_{-\infty}^{\infty} f(x,y)\,dx\,dy$$

$$\Rightarrow \int_{-\infty}^{b}\left(\int_{-\infty}^{\infty} f(x,y)\,dx\right)dy = \int_{-\infty}^{b} f_Y(y)\,dy$$

Now, differentiating both sides of the above equn. w.r.t (y), we've:

$$\int_{-\infty}^{\infty} f(x,y)\,dx = f_Y(y)$$

$$\therefore \boxed{f_Y(y) = \int_{-\infty}^{\infty} f(x,y)\,dx}$$

Q: The joint PDF of (X) & (Y) is given by:

$$f(x,y) = \begin{cases} 2e^{-x}\cdot e^{-2y}, & 0<(x,y)<\infty \\ 0, & \text{otherwise} \end{cases}$$

compute, ⓐ $P\{X>1, Y<1\}$ ⓑ $P\{X<Y\}$ ⓒ $P\{X<a\}$

Ans- ⓐ $P\{X>1, Y<1\} = \int_{-\infty}^{1}\int_{1}^{\infty} f(x,y)\,dx\,dy$

$$= \int_{0}^{1}\int_{1}^{\infty} 2e^{-x}\cdot e^{-2y}\,dx\,dy = 2\int_{0}^{1}\left(\int_{1}^{\infty} e^{-x}\,dx\right)e^{-2y}\,dy$$

$$= 2\int_{0}^{1}\left[-e^{-x}\right]_{1}^{\infty}\cdot e^{-2y}\,dy = 2\int_{0}^{1}(-0+e^{-1})\,e^{-2y}\,dy$$

$$= 2e^{-1}\int_{0}^{1} e^{-2y}\,dy = \cancel{2}e^{-1}\left[\frac{-e^{-2y}}{\cancel{2}}\right]_{0}^{1} = e^{-1}(-e^{-2}+1)$$

$$= -e^{-3}+e^{-1} \text{ (Ans) } = 0.318$$

Math

ⓑ $P\{X<Y\} = P\{X<Y, Y<\infty\}$

$$= \int_{-\infty}^{\infty}\int_{\infty}^{y} f(x,y)\,dx\,dy = \int_{0}^{\infty}\int_{0}^{y} 2e^{-x}\cdot e^{-2y}\,dx\,dy$$

$$= 2\int_{0}^{\infty}\left(\int_{0}^{y} e^{-x}\,dx\right)\cdot e^{-2y}\,dy$$

$$= 2\int_{0}^{\infty}\left[-e^{-x}\right]_{0}^{y}\cdot e^{-2y}\,dy = 2\int_{0}^{\infty}\left(-e^{-y}+1\right)\cdot e^{-2y}\,dy$$

$$= 2\int_{0}^{\infty}\left(-e^{-3y}+e^{-2y}\right)dy = 2\left[\frac{-e^{-3y}}{-3}+\frac{e^{-2y}}{-2}\right]_{0}^{\infty}$$

$$= -2\left[\frac{-e^{-3y}}{3}+\frac{e^{-2y}}{2}\right]_{0}^{\infty} = -2\left[\left(0+\frac{1}{3}\right)+\left(0-\frac{1}{2}\right)\right]$$

$$= -2\left[\frac{1}{3}-\frac{1}{2}\right] = \frac{1}{3}$$

ⓒ $P\{X<a\} = P\{X<a, Y<\infty\}$

$$= \int_{-\infty}^{\infty}\int_{-\infty}^{a} f(x,y)\,dx\,dy = \int_{0}^{\infty}\int_{0}^{a} 2e^{-x}\cdot e^{-2y}\,dx\,dy$$

$$= 2\int_{0}^{\infty}\left(\int_{0}^{a} e^{-x}\,dx\right)\cdot e^{-2y}\,dy = 2\int_{0}^{\infty}\left[-e^{-x}\right]_{0}^{a}\cdot e^{-2y}\,dy$$

$$= 2\int_{0}^{\infty}\left(-e^{-a}+1\right)\cdot e^{-2y}\,dy = 2\int_{0}^{\infty}\left(-e^{-a}\cdot e^{-2y}+e^{-2y}\right)dy$$

$$= 2\left[-e^{-a}\int_{0}^{\infty} e^{-2y}\,dy + \int_{0}^{\infty} e^{-2y}\,dy\right] = 2(1-e^{-a})\int_{0}^{\infty} e^{-2y}\,dy$$

$$= 2(1-e^{-a})\left[\frac{e^{-2y}}{-2}\right]_{0}^{\infty} = \cancel{2}(1-e^{-a})\left[0+\frac{1}{2}\right] = 1-e^{-a}$$

11→ Independent Random Variables:-

N.B.:
$$\boxed{\begin{array}{l} P(A\cap B)=\phi \rightarrow \text{Mutually Exclusive.} \\ P(A\cap B)=P(A)\cdot P(B) \rightarrow \text{Independent Event.} \end{array}}$$

- 2 random variables (X) & (Y) are said to be independent if for any 2 sets of real nos. (A) & (B), we've:

$$\boxed{P\{X\in A, Y\in B\}=P\{X\in A\}\cdot P\{Y\in B\}}$$

-OR-

$$\boxed{P\{X\leq a, Y\leq b\}=P\{X\leq a\}\cdot P\{Y\leq b\}}$$

- (X) & (Y) are called independent if for (A) & (B), the events $E_A=\{X\in A\}$ & $E_B=\{Y\in B\}$ are independent.
- In case of joint CDF, (F) of (X) & (Y), i.e. F_{XY}, (X) & (Y) are independent if:

$$\boxed{F_{XY}(x,y)=F_X(x)\cdot F_Y(y)}$$

- **~~Discrete~~ Continuous case:-**

  In case of joint PDF (f_{XY}) of (X) & (Y), (X) & (Y) are called independent if:

$$\boxed{f_{XY}(x,y)=f_X(x)\cdot f_Y(y)}$$

- **Discrete case:-**

  In case of joint PMF (P_{XY}) of (X) & (Y), (X) & (Y) are called independent if:

$$\boxed{P_{XY}(x,y)=P_X(x)\cdot P_Y(y)}$$

Remark: (X) & (Y) are independent if knowing the value of 1 doesn't change the distribut^{n} of other.

- Random variables that aren't independent are called dependent.

II → Mean & Variance of (X) & (Y) in a Bivariate distribut^n:-

- $E[X] = \sum x P_X(x)$ (Discrete)

  $= \int_{-\infty}^{\infty} x f(x)\,dx$ (Continuous)

- $E[Y] = \sum y P_y(y)$ (Discrete)

  $= \int_{-\infty}^{\infty} y f(y)\,dy$ (Continuous)

- $Var(X) = E[X^2] - (E[X])^2$
- $Var(Y) = E[Y^2] - (E[Y])^2$

Q: The joint PDF of (X) & (Y) is given by:

$$f(x,y) = \begin{cases} e^{-(x+y)}, & 0<(x,y)<\infty \\ 0, & \text{otherwise} \end{cases}$$

Then, find the PDF of X/Y

Ans- $F_{X/Y}(a) = P\left\{\frac{X}{Y} \le a\right\} = P\{X \le aY, Y<\infty\}$

$$= \int_0^{\infty}\int_0^{ay} f(x,y)\,dx\,dy = \int_0^{\infty}\int_0^{ay} e^{-(x+y)}\,dx\,dy$$

$$= \int_0^{\infty} \left[-e^{-x}\right]_0^{ay} \cdot e^{-y}\,dy = 1 - \frac{1}{1+a}$$

Now, differentiating both sides w.r.t (a), we've: $\boxed{\frac{d}{dx}[CDF] = PDF} \Rightarrow \boxed{\int PDF = CDF}$

$$\frac{d}{da} F_{X/Y}(a) = \frac{d}{da}\left(1 - \frac{1}{1+a}\right) \Rightarrow f_{X/Y}(a) = \frac{1}{(1+a)^2}$$

Q: The joint PDF of (X) & (Y) is given by:

$$f(x,y) = \begin{cases} x+y, & 0<(x,y)<1 \\ 0, & \text{otherwise} \end{cases}$$

- Are (X) & (Y) independent
- Find the marginal PDF of (X).
- Find $P\{X+Y<1\}$

Ans- ⓐ $f_X(x) = \int_0^1 f(x,y)\,dy = \int_0^1 (x+y)\,dy$

$= x + \frac{1}{2}$

$f_Y(y) = \int_0^1 f(x,y)\,dx = \int_0^1 (x+y)\,dx$

$= y + \frac{1}{2}$

$\therefore f(x,y) = x+y \neq \left(x+\frac{1}{2}\right)\cdot\left(y+\frac{1}{2}\right) = f_X(x)\cdot f_Y(y)$

So, (X) & (Y) aren't independent.

ⓑ $f_X(x) = x + \frac{1}{2}, \quad 0<x<1$

ⓒ $P\{X+Y<1\} = P\{X<1-Y, Y<\infty\}$

$= \int_0^1\int_0^{1-y} f(x,y)\,dx\,dy = \int_0^1\int_0^{1-y} (x+y)\,dx\,dy$

$= \int_0^1\left(\int_0^{1-y} (x+y)\,dx\right)dy = \int_0^1\left[\frac{x^2}{2}+xy\right]_0^{1-y} dy$

$= \int_0^1\left(\frac{(1-y)^2}{2}+y-y^2\right)dy = 0.33$

Q: The random variables (X) & (Y) have joint PDF $f(x,y) = \begin{cases} 12xy(1-x), & 0<x,y<1 \\ 0, & \text{otherwise} \end{cases}$

ⓐ Are (X) & (Y) independent

ⓑ Find E[X]

ⓒ Find E[Y]

ⓓ Var(X) ⓔ Var(Y)

Ans- ⓐ $f_X(x) = \int_0^1 f(x,y)\,dy = \int_0^1 12xy(1-x)\,dy$

$= 6x(1-x)$

$$f_Y(y) = \int_0^1 f(x,y)\,dx = \int_0^1 12xy(1-x)\,dx$$

$$= 2y$$

$$\therefore f_{XY}(x,y) = f_X(x)\cdot f_Y(y) = 6x(1-x) \times 2y$$
$$= 12xy(1-x)$$

$\therefore$ (X) & (Y) are independent

ⓑ $$E[X] = \int_{-\infty}^{\infty} x f_X(x)\,dx = \int_0^1 6x^2(1-x)\,dx = \frac{1}{2}$$

$$E[Y] = \int_{-\infty}^{\infty} y f_Y(y)\,dy = \int_0^1 y \cdot 2y\,dy = \frac{2}{3}$$

$$E[X^2] = \int_{-\infty}^{\infty} x^2 f_X(x)\,dx = \int_0^1 6x^3(1-x)\,dx = \frac{3}{10}$$

$$E[Y^2] = \int_{-\infty}^{\infty} y^2 f_Y(y)\,dy = \int_0^1 y^2 \cdot 2y\,dy = \frac{1}{2}$$

ⓒ $\operatorname{var}(X) = E[X^2] - (E[X])^2 = \frac{1}{20}$

ⓓ $\operatorname{var}(Y) = E[Y^2] - (E[Y])^2 = \frac{1}{18}$